

NEW CRITERION FOR ALGEBRAIC VOLUME DENSITY PROPERTY

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ABSTRACT. A smooth affine algebraic variety X equipped with an algebraic volume form ω has the algebraic volume density property (AVDP) if the Lie algebra generated by completely integrable algebraic vector fields of ω -divergence zero coincides with the space of all algebraic vector fields of ω -divergence zero. We develop an effective criterion of verifying whether a given X has AVDP. As an application of this method we establish AVDP for any homogeneous space $X = G/R$ that admits a G -invariant algebraic volume form where G is a linear algebraic group and R is a closed reductive subgroup of G .

1. INTRODUCTION

In 1990's Andersén and Lempert [1], [2] discovered a remarkable property of complex Euclidean spaces that to a great extent compensates for the lack of partition of unity for holomorphic automorphisms. It is called the density property (this terminology was introduced later by Varolin [22]) or for short DP. A Stein manifold X has DP if the Lie algebra generated by completely integrable holomorphic vector fields is dense (in the compact-open topology) in the space of all holomorphic vector fields on X . In the presence of DP one can construct global holomorphic automorphisms of X with prescribed local properties. More precisely, any local phase flow on a Runge domain in X can be approximated by global automorphisms. Needless to say that this lead to remarkable consequences (see [7], [21], [22], [23], [13]).

If X is equipped with a holomorphic volume form ω (i.e. ω is a nowhere vanishing top holomorphic differential form) then one can ask whether a similar approximation holds for automorphisms and phase flows preserving ω . Under a mild additional assumption the answer is yes in the presence of the volume density property (VDP) which means that the Lie algebra generated by completely integrable holomorphic vector fields of ω -divergence zero is dense in the space of all holomorphic vector fields of ω -divergence zero.

2000 *Mathematics Subject Classification.* Primary: 32M05, 14R20. Secondary: 14R10, 32M25.

Key words and phrases. affine space.

Acknowledgements: This research was started during a visit of the first author to the University of Bern and continued during a visit of the second author to the University of Miami, Coral Gables. We thank these institutions for their generous support and excellent working conditions. The research of the first author was also partially supported by NSA Grant no. H982301010185 and the second author was also partially supported by Schweizerische Nationalfonds grant No. 200020-134876/1.

In this paper we study the case when X is a smooth affine algebraic variety (over \mathbb{C}) and whenever a volume form ω is present it is an algebraic one. The following definitions are due to Varolin and the authors.

1.1. Definition. We say that X has the algebraic volume density property (ADP) if the Lie algebra $\text{Lie}_{\text{alg}}(X)$ generated by the set $\text{IVF}(X)$ of completely integrable algebraic vector fields coincides with the space $\text{AVF}(X)$ of all algebraic vector fields on X . Similarly in the presence of ω we can speak about the algebraic volume density property (AVDP) that means the equality $\text{Lie}_{\text{alg}}^{\omega}(X) = \text{AVF}_{\omega}(X)$ for analogous objects (that is, all participating vector fields have ω -divergence zero; say $\text{Lie}_{\text{alg}}^{\omega}(X)$ is generated by $\text{IVF}_{\omega}(X)$).

It is worth mentioning that ADP and AVDP imply DP and VDP respectively (where the second implication is not that obvious) and in particular all remarkable consequences for complex analysis on X .

An effective criterion whether X has ADP was developed by the authors in [11]. The main idea was to search for a nontrivial $\mathbb{C}[X]$ -module inside $\text{Lie}_{\text{alg}}(X)$. It lead in particular to the proof of ADP for almost all homogeneous spaces of form G/R where G is any linear algebraic group and R is a closed reductive subgroup of G [5] (one needs to have R reductive for G/R to be affine).

At first glance this idea does not work in the volume-preserving case. Indeed, $\text{Lie}_{\text{alg}}^{\omega}(X)$ cannot contain a $\mathbb{C}[X]$ -module by the following reason. If $\nu \in \text{AVF}_{\omega}(X)$ then for $f \in \mathbb{C}[X]$ the divergence of the vector field $f\nu$ is computed by formula $\text{div}_{\omega}(f\nu) = \nu(f)$, i.e. it is nonzero for a general f .

However in this paper we establish a criterion for the volume-preserving case whose effectiveness is comparable with the one in [11] for ADP. It turns out that the main idea from [11] works: we do need to catch a $\mathbb{C}[X]$ -module but in a space different from $\text{Lie}_{\text{alg}}^{\omega}(X)$. To describe this space we need some extra notation. Let $\mathcal{C}_k(X)$ be the space of algebraic differential k -forms on X and $\mathcal{Z}_k(X)$ and $\mathcal{B}_k(X)$ be its subspaces of closed and exact k -forms respectively. If $\dim X = n$ then there exists an isomorphism $\Theta : \text{AVF}_{\omega}(X) \rightarrow \mathcal{Z}_{n-1}(X)$ given by the formula $\xi \rightarrow \iota_{\xi}\omega$ where $\iota_{\xi}\omega$ is the interior product of ω and $\xi \in \text{AVF}_{\omega}(X)$. Consider the homomorphism $D_k : \mathcal{C}_{k-1}(X) \rightarrow \mathcal{B}_k(X)$ generated by outer differentiation d and let $D = D_{n-1}$. The main theme of our new criterion is the search for a $\mathbb{C}[X]$ -module in the space $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^{\omega}(X))$. With some additional assumptions the existence of such a module implies AVDP. The situation is in fact even more pleasant because the methods we developed earlier in [11] and [5] for checking whether $\text{Lie}_{\text{alg}}(X)$ contains a $\mathbb{C}[X]$ -module work perfectly for the space $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^{\omega}(X))$.

The absence of such a simple approach was the main source of difficulties in [14] where, in particular, we proved AVDP for all linear algebraic groups with respect to left (or right) invariant volume forms. Now we are able not only to demonstrate a drastic simplification of the proof of this result but to establish AVDP for every homogeneous space G/R as before provided this space is equipped with a G -invariant volume form (actually, we prove a stronger statement, see Theorem 4, Remark 6.3, and Corollary 6.4 below).

2. PRELIMINARIES

The main aim of this section is to remind the definition and some properties of semi-compatible vector fields. This notion will be our main tool in the search of $\mathbb{C}[X]$ -modules in the space $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X))$ discussed in the Introduction.

2.1. Notation. In what follows X is always a smooth affine algebraic variety over \mathbb{C} (except for Remark 5.2 where it is normal) and all other notations mentioned in Introduction will remain valid. We consider often a situation when X is equipped also with an algebraic action of a finite group Γ . We call such an X a Γ -variety. The ring of regular Γ -invariant functions will be denoted by $\mathbb{C}[X, \Gamma]$; it is naturally isomorphic to the ring $\mathbb{C}[X/\Gamma]$ of regular functions on the quotient space X/Γ .

2.2. Definition. Let ξ and η be nontrivial completely integrable algebraic vector fields on a Γ -variety X which are Γ -invariant. We say that the pair (ξ, η) is Γ -semi-compatible

if the span of $(\text{Ker } \xi \cap \mathbb{C}(X, \Gamma)) \cdot (\text{Ker } \eta \cap \mathbb{C}(X, \Gamma))$ contains a nonzero ideal of $\mathbb{C}[X, \Gamma]$.

The largest ideal contained in the span will be called the associate Γ -ideal of the pair (ξ, η) . In the case of a trivial Γ we say that the pair (ξ, η) is semi-compatible. In this terminology Γ -semi-compatibility of (ξ, η) follows from semi-compatibility of (ξ', η') where ξ' and η' are the vector fields on X/Γ induced by ξ and η .

Recall also that an algebraic \mathbb{C}_+ -action (resp. \mathbb{C}^* -action) on X may be always viewed as the phase flow of an algebraic vector field that is called locally nilpotent (resp. semi-simple). Semi-compatible pairs (ξ, η) such that ξ and η are either locally nilpotent or semi-simple will play a crucial role in this paper.

2.3. Example. (1) Let X_i ($i = 1, 2$) be an affine algebraic Γ_i -variety and $p_i : X := X_1 \times X_2 \rightarrow X_i$ be the natural projection. Let Γ_i and $\Gamma_1 \times \Gamma_2$ act naturally on X . Suppose that ξ_i is Γ_i -invariant completely integrable algebraic vector field on X such that $(p_j)_*(\xi_i) = 0$ for $1 \leq i \neq j \leq 2$. Since $(p_i)^*(\mathbb{C}[X_i, \Gamma_i]) \subset (p_i)^*(\mathbb{C}[X_i]) \subset \text{Ker } \xi_j$ we see that (ξ_1, ξ_2) is a $(\Gamma_1 \times \Gamma_2)$ -semi-compatible pair on X whose associate ideal is $\mathbb{C}[X, \Gamma_1 \times \Gamma_2]$. Furthermore, let Ξ_i be the set of irreducible characters of representations for Γ_i . Then $\mathbb{C}[X_i] = \bigoplus_{\chi_i \in \Xi_i} U_i(\chi_i)$ where $U_i(\chi_i)$ is the subspace of $\mathbb{C}[X_i]$ associated with character χ_i . Hence $\mathbb{C}[X] = \bigoplus_{(\chi_1, \chi_2) \in \Xi_1 \times \Xi_2} U_1(\chi_1) \otimes U_2(\chi_2)$. This implies that for every subgroup Γ of $\Gamma_1 \times \Gamma_2$ the space $\mathbb{C}[X, \Gamma] \subset \mathbb{C}[X]$ is the direct sum of spaces $U_1(\chi_1) \otimes U_2(\chi_2)$ for which the restriction of the character $\chi_1 \cdot \chi_2$ to Γ is the trivial character. Since $U_1(\chi_1) \otimes U_2(\chi_2) \subset \text{Ker } \xi_1 \times \text{Ker } \xi_2$ we see that (ξ_1, ξ_2) is also a Γ -semi-compatible pair on X whose associate ideal is $\mathbb{C}[X, \Gamma]$.

(2) Consider $X = SL_2$ as a subvariety of $\mathbb{C}_{a_1, a_2, b_1, b_2}^4$ given by $a_1 b_2 - a_2 b_1 = 1$ (where by $\mathbb{C}_{z_1, \dots, z_n}^n$ we denote a Euclidean space \mathbb{C}^n with a fixed coordinate system (z_1, \dots, z_n)). The vector fields

$$\xi = b_1 \partial / \partial a_1 + b_2 \partial / \partial a_2 \quad \text{and} \quad \eta = a_1 \partial / \partial b_1 + a_2 \partial / \partial b_2$$

are locally nilpotent on X and if we present every $A \in X$ as a matrix

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

then ξ (resp. η) is associated with left multiplication by elements of the upper (resp. lower) triangular unipotent \mathbb{C}_+ -subgroup U (resp. L) of SL_2 . Note that $\mathbb{C}[b_1, b_2] \subset \text{Ker } \xi$ and $\mathbb{C}[a_1, a_2] \subset \text{Ker } \eta$ which implies that the pair (ξ, η) is semi-compatible and the associate ideal is again $\mathbb{C}[X]$. If I is the identity matrix and Γ is the subgroup $\{I, -I\}$ of SL_2 then ξ and η induce locally nilpotent vector fields ξ' and η' on $Y = X/\Gamma \simeq PSL_2$. The fact that the pair (ξ', η') is semi-compatible is less trivial but true by the Proposition 2.4 below.

2.4. Proposition. ([11][Lemma 3.6]) *Let X be a Γ -variety, ξ and η be semi-compatible algebraic vector fields. Let ξ be locally nilpotent, η be either locally nilpotent or semi-simple, and one of the following conditions hold*

- (i) $[\xi, \eta] = 0$ or
- (ii) η is also locally nilpotent and the Lie algebra generated by ξ and η is isomorphic to \mathfrak{sl}_2 .

Suppose that ξ and η are Γ -invariant and that ξ' and η' are the induced vector fields on X/Γ . Then the pair (ξ', η') is semi-compatible, i.e. (ξ, η) is Γ -semi-compatible.

The fact that locally nilpotent vector fields ξ and η generate a Lie algebra isomorphic to \mathfrak{sl}_2 does not imply in general that they are semi-compatible. However we have the following.

2.5. Proposition ([5][Theorem 12]). *Let X admit a fixed point free non-degenerate SL_2 -action (i.e. general SL_2 -orbits are of dimension 3) and let U (resp. L) be the unipotent subgroup of upper (resp. lower) triangular matrices in SL_2 as in Example 2.3 (2). Suppose that ξ and η are the locally nilpotent vector fields associated with the induced actions of the \mathbb{C}_+ -groups U and L on X . Then (ξ, η) is a semi-compatible pair.*

2.6. Remark. The assumption that the action is fixed point free and non-degenerate is essential for the validity of Proposition 2.5. In the presence of fixed points the statement does not hold (see [4]) and for the degenerate case SL_2/\mathbb{C}^* provides a counterexample because of the following.

2.7. Proposition. *Let X be a smooth affine surface different from \mathbb{C}^2 , $\mathbb{C}^* \times \mathbb{C}^*$, or $\mathbb{C}^* \times \mathbb{C}$. Then X does not admit a semi-compatible pair of algebraic vector fields that are locally nilpotent or semi-simple.*

Proof. Assume that ξ_1 and ξ_2 are such vector fields on X . Let H_i be the \mathbb{C}_+ or \mathbb{C}^* group associated with ξ_i acting on X and let $\rho_i : X \rightarrow X_i := X/H_i$ be the quotient morphism. If X_1 is a point or if it is a curve different from \mathbb{C} and \mathbb{C}^* then X_1 does not admit nonconstant homomorphisms from \mathbb{C} or \mathbb{C}^* and therefore any general orbit of H_2 must be contained in a fiber of ρ_1 . Hence morphism $\rho = (\rho_1, \rho_2) : X \rightarrow X_1 \times X_2$ is not birational contrary to the fact that semi-compatibility is equivalent to the claim that $Z = \rho(X)$ is closed in $X_1 \times X_2$ and ρ is finite birational by Proposition 3.4 in [11].

Thus $Z = X_1 \times X_2$ is one of the surfaces listed in the statement of Proposition and the Zariski Main theorem implies that X is isomorphic Z . \square

The existence of fixed point free non-degenerate SL_2 actions on homogeneous spaces is guaranteed by the next result.

2.8. Proposition ([5][Theorem 24]). *Let G be a semi-simple group and R be a proper closed reductive subgroup of G such that $X = G/R$ is at least three-dimensional. Then there exists an SL_2 subgroup of G whose natural action on X is fixed point free and non-degenerate. In particular, X admits semi-compatible pairs of locally nilpotent vector fields.*

Besides the properties of semi-compatible pair listed above we need also the following well-known consequence of the Nakayama lemma.

2.9. Lemma. *Let A be an affine domain and M be an A -module equipped with a filtration $M = \bigcup_{i \in \mathbb{N}} M_i$, $M_i \subseteq M_{i+1}$ such that each M_i is a finitely generated module over A . Suppose that N is a submodule of M such that for every maximal ideal μ of A one has $M/\mu M = N/\mu N$. Then $M = N$.*

Proof. It suffices to show that $M_i = N_i$ for every $i \in \mathbb{N}$ where $N_i = N \cap M_i$, i.e. we can suppose that M itself is finitely generated. Furthermore, the Nullstellensatz implies that $M = N$ iff the localized versions of this equality with respect to every μ are valid. Thus it is enough to consider the case when A is a local ring with the only maximal ideal μ . For the A -module $L = M/N$ the equality $M/\mu M = N/\mu N$ implies that $L = \mu L$, i.e. by the Nakayama lemma L is the zero module which is the desired conclusion. \square

3. THE CRITERION

3.1. Notation. Suppose that X is a Γ -variety equipped with an algebraic volume form ω . Then one has $\omega \circ \gamma = \chi_\omega(\gamma)\omega$ for every $\gamma \in \Gamma$ where $\chi_\omega : \Gamma \rightarrow \mathbb{C}^*$ is a character. We denote by $\text{Lie}_{\text{alg}}^\omega(X, \Gamma)$ the Lie algebra generated by Γ -invariant completely integrable algebraic vector fields of ω -divergence zero. Similarly $\text{AVF}_\omega(X, \Gamma)$ will be the space of algebraic Γ -invariant divergence-free vector fields. Such fields can be also viewed as elements of $\text{AVF}_\omega(X/\Gamma)$. By $\mathcal{C}_k(X, \Gamma)$ we denote the space of Γ -quasi-invariant algebraic k -forms corresponding to the character χ_ω . By $\mathcal{Z}_k(X, \Gamma)$ and $\mathcal{B}_k(X, \Gamma)$ we denote the subspaces of closed and exact forms in $\mathcal{C}_k(X, \Gamma)$. Recall that

$$\Theta : \text{AVF}_\omega(X) \rightarrow \mathcal{Z}_{n-1}(X)$$

is the isomorphism given by the formula $\xi \rightarrow \iota_\xi \omega$ (where $\iota_\xi \omega$ is the interior product of ω and $\xi \in \text{AVF}_\omega(X)$) and $D_k : \mathcal{C}_{k-1}(X) \rightarrow \mathcal{B}_k(X)$ is the homomorphism generated by outer differentiation d . In these notations the restriction of Θ generates an isomorphism between $\text{AVF}_\omega(X, \Gamma)$ and $\mathcal{Z}_k(X, \Gamma)$ while D_k sends $\mathcal{C}_{k-1}(X, \Gamma)$ into $\mathcal{B}_k(X, \Gamma)$.

3.2. Lemma. *In Notation 3.1 one has $\mathcal{B}_k(X, \Gamma) = D_k(\mathcal{C}_{k-1}(X, \Gamma))$ for every $k \geq 1$.*

Proof. Let Ξ be the set of irreducible characters of representations of Γ . Then $\mathcal{B}_k(X) = \bigoplus_{\chi \in \Xi} V_\chi$ and $\mathcal{C}_{k-1}(X) = \bigoplus_{\chi \in \Xi} U_\chi$ where V_χ and U_χ are the subspaces corresponding to character χ . Since $D_k(U_\chi) \subset V_\chi$ and $\mathcal{B}_k(X) = D_k(\mathcal{C}_{k-1}(X))$ one has $D_k(U_\chi) = V_\chi$ which implies the desired conclusion. \square

The next simple observation provides a crucial connection between semi-compatibility and existence of $\mathbb{C}[X]$ -modules in $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ where $D = D_{n-1}$.

3.3. Proposition. *Let ξ and η be vector fields from $\text{AVF}_\omega(X)$.¹ Then*

$$(1) \quad \iota_{[\xi, \eta]} \omega = d\iota_\xi \iota_\eta \omega.$$

Proof. Recall the following relations between the outer differentiation d , Lie derivative L_ξ and the interior product ι_ξ [15]

$$(2) \quad L_\xi = d\iota_\xi + \iota_\xi d \quad \text{and} \quad [L_\xi, \iota_\eta] = \iota_{[\xi, \eta]}.$$

By this formula

$$\iota_\xi d\iota_\eta \omega = \iota_\xi (L_\eta - \iota_\eta d) \omega$$

where the right-hand side is zero since $L_\eta \omega - \iota_\eta d\omega = 0$ for closed ω and η of ω -divergence zero. Then another application of formula (2) in combination with the fact that $\iota_\xi d\iota_\eta \omega = 0$ yields

$$[L_\xi, \iota_\eta] \omega = L_\xi \iota_\eta \omega - \iota_\eta L_\xi \omega = L_\xi \iota_\eta \omega = d\iota_\xi \iota_\eta \omega + \iota_\xi d\iota_\eta \omega = d\iota_\xi \iota_\eta \omega.$$

Thus by formula (2) we have the desired equality

$$\iota_{[\xi, \eta]} \omega = d\iota_\xi \iota_\eta \omega. \quad \square$$

3.4. Corollary. *Let X be a Γ -variety equipped with an algebraic volume form ω and let ξ and η be Γ -semi-compatible vector fields on X . Suppose that $\Theta(\xi)$ and $\Theta(\eta)$ are contained in $\mathcal{B}_{n-1}(X)$ ² which is always the case when $H^{n-1}(X, \mathbb{C}) = 0$. Then $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains a nontrivial $\mathbb{C}[X, \Gamma]$ -submodule L of the module $\mathcal{C}_{n-2}(X, \Gamma)$.*

Proof. Formula (1) from Proposition 3.3 implies that for $f \in \text{Ker } \xi$, $g \in \text{Ker } \eta$, and $\kappa = [f\xi, g\eta]$ one has $\iota_\kappa \omega = d\iota_{f\xi} \iota_{g\eta} \omega$, i.e. $(fg)\iota_\xi \iota_\eta \omega \in D^{-1}(\iota_\kappa \omega)$. Choose Γ -invariant f and g . Then by Γ -semi-compatibility $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X))$ contains $\mathbb{C}[X, \Gamma]$ -module $I\iota_\xi \iota_\eta \omega$ where I is a nontrivial ideal of $\mathbb{C}[X, \Gamma]$ which is the desired conclusion. \square

¹Actually the statement remains valid when X is a complex manifold equipped with a holomorphic volume form ω and ξ and η are two holomorphic vector fields on X of ω -divergence zero.

²Since $\Theta^{-1}(\mathcal{Z}_{n-1}(X)) = \text{AVF}_\omega(X)$ this condition implies that ξ and η have ω -divergence zero. Actually if any of these fields is locally nilpotent it is automatically divergence-free without this assumption.

3.5. Notation. Suppose that X is an n -dimensional Γ -variety equipped with a volume form ω . For any subbundle V of $\Lambda^k TX$ we denote by $\mathcal{C}_{n-k}^V(X, \Gamma)$ the “dual” subbundle that consists of all forms τ such that for every $x \in X$ the restriction of τ to $\Lambda^{n-k} T_x X$ is contained in the space generated by elements $\iota_{v_1} \dots \iota_{v_k} \omega(x)$ where $v_1 \wedge \dots \wedge v_k$ runs over V_x .

3.6. Proposition. *Let X be a Γ -variety equipped with an algebraic volume form ω and let $(\xi_j, \eta_j)_{j=1}^k$ be pairs of Γ -semi-compatible vector fields such that $\Theta(\xi_j), \Theta(\eta_j) \in \mathcal{B}_{n-1}(X)$ for every j . Let I_j be the Γ -ideal associated with (ξ_j, η_j) , and let $I_j(x) = \{f(x) | f \in I_j\}$ for $x \in X$. Suppose that V is a subbundle of $\Lambda^2 TX$ such that*

(A) *for every $x \in X$ the set $\{I_j(x)\xi_j(x) \wedge \eta_j(x)\}_{j=1}^k$ generates the fiber V_x of V over x .*

Then $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains $D(\mathcal{C}_{n-2}^V(X, \Gamma))$. In particular, if $V = \Lambda^2 TX$ (in which case we say that the strong Condition (A) holds) then $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains $\mathcal{B}_{n-1}(X, \Gamma)$.

Proof. By Corollary 3.4 $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains a $\mathbb{C}[X, \Gamma]$ -module L of form

$$L = \sum_{j=1}^k I_j \iota_{\xi_j} \iota_{\eta_j} \omega.$$

Treat now $\mathcal{C}_{n-2}^V(X, \Gamma)$ as a $\mathbb{C}[X/\Gamma]$ -module, and L as its submodule. By Lemma 2.9 the module $\mathcal{C}_{n-2}^V(X, \Gamma)$ coincides with L if for every $x \in X/\Gamma$ one has

$$\mathcal{C}_{n-2}^V(X, \Gamma)/(\mu(x)\mathcal{C}_{n-2}^V(X, \Gamma)) = L/(\mu(x)L)$$

where $\mu(x)$ is the maximal ideal in $\mathbb{C}[X/\Gamma]$ associated with the point x . The last equality is, of course, equivalent to Condition (A). Thus we have the desired conclusion. \square

3.7. Corollary. *Let X be a Γ -variety equipped with an algebraic volume form ω and let $(\xi_j, \eta_j)_{j=1}^{k_0}$ be pairs of Γ -semi-compatible vector fields such that $\Theta(\xi_j), \Theta(\eta_j) \in \mathcal{B}_{n-1}(X)$ for every j . Suppose that F is a group of algebraic automorphisms of X with the following properties:*

- (i) *the natural F -action commutes with the Γ -action and preserves ω up to constant factors;*
- (ii) *F induces a transitive action on X/Γ ³.*

Suppose that V is the subbundle of $\Lambda^2 TX$ such that it is invariant under the natural F -action and at a general point $x_0 \in X$ its fiber V_{x_0} is generated by the set $\{\xi_j(x_0) \wedge \eta_j(x_0)\}_{j=1}^{k_0}$. Then $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains $D(\mathcal{C}_{n-2}^V(X, \Gamma))$.

Proof. Since x_0 is a general point $I_j(x_0) = \{f(x_0) | f \in I_j\} \neq 0$ for every j where I_j is the Γ -ideal associate with (ξ_j, η_j) . In particular, $\{I_j(x_0)\xi_j(x_0) \wedge \eta_j(x_0)\}_{j=1}^{k_0}$ generates V_{x_0} . Since every automorphism $\alpha \in F$ sends ω into $c\omega$, $c \in \mathbb{C}^*$, it preserves the set of

³Condition (ii) implies that the Γ -action is free.

vector fields with ω -divergence zero. Complete integrability is also preserved by α . In combination with transitivity of the F -action on X/Γ this implies that we can extend the sequence to $(\xi_j, \eta_j)_{j=1}^{k_0}$ to a larger sequence of pairs of semi-compatible divergence-free vector fields $(\xi_j, \eta_j)_{j=1}^k$ for which Condition (A) from Proposition 3.6 holds. Hence we are done. \square

As a consequence of Proposition 3.6 we have our main criterion.

Theorem 1. *Let the assumption of Proposition 3.6 hold. Suppose also that the following condition is true*

(B) *the image of $\Theta(\text{Lie}_{\text{alg}}^\omega(X))$ under De Rham homomorphism $\Phi_{n-1} : \mathcal{Z}_{n-1}(X) \rightarrow H^{n-1}(X, \mathbb{C})$ coincides with the subspace $\Phi_{n-1}(\mathcal{Z}_{n-1}(X, \Gamma))$ of $H^{n-1}(X, \mathbb{C})$.*

Then $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma)) = \mathcal{Z}_{n-1}(X, \Gamma)$ and therefore $\text{Lie}_{\text{alg}}^\omega(X, \Gamma) = \text{AVF}_\omega(X, \Gamma)$.

3.8. Definition. (1) If X is a Γ -variety equipped with an algebraic volume form ω and the equality $\text{Lie}_{\text{alg}}^\omega(X, \Gamma) = \text{AVF}_\omega(X, \Gamma)$ holds then we say that X has Γ -AVDP (with respect to ω).

(2) If under the assumption of Theorem 1 the group Γ is trivial (or if it acts trivially) then we say that the pair (X, ω) satisfies Condition (B). In the case of a nontrivial Γ -action we speak about the validity of Condition (B) for the triple (X, ω, Γ) .

4. CONDITION (B)

Actually, in this section we are checking at first glance a stronger condition that the image of $\Theta(\text{IVF}_\omega(X))$ under De Rham homomorphism $\Phi_{n-1} : \mathcal{Z}_{n-1}(X) \rightarrow H^{n-1}(X, \mathbb{C})$ coincides with $H^{n-1}(X, \mathbb{C})$. However it is equivalent to Condition (B) by the following proposition.

4.1. Proposition. *Let X and ω be as in Definition 3.1 and N be the subspace of $\text{Lie}_{\text{alg}}^\omega(X)$ generated by elements that can be presented as Lie brackets of other elements. Then*

(1) *$\Theta(N)$ is contained in the kernel of Φ_{n-1} ,*

(2) *the image of $\Theta(\text{IVF}_\omega(X))$ under Φ_{n-1} generates $H^{n-1}(X, \mathbb{C})$ provided pair (X, ω) satisfies Condition (B).*

Proof. The second statement is a consequence of the first one which in turn follows from Formula (1). \square

4.2. Lemma. *Let T be a torus $(\mathbb{C}^*)^n$. Then Condition (B) is valid for an invariant volume form on T .*

Proof. Equip $T = (\mathbb{C}^*)^n$ with a natural coordinate system (z_1, \dots, z_n) , i.e. an invariant volume form can be chosen as $\omega = \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_n}{z_n}$. By the Künneth formula $H^{n-1}(T, \mathbb{C}) \simeq$

\mathbb{Z}^n . Consider the semi-simple vector fields $\nu_i = z_i \frac{\partial}{\partial z_i}$, i.e. ν_i is associated with the \mathbb{C}^* -action $(z_1, \dots, z_i, \dots, z_n) \rightarrow (z_1, \dots, \lambda z_i, \dots, z_n)$ preserving ω . Hence ν_i has the ω -divergence zero. It remains to note that

$$\iota_{\nu_i} \omega = \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_{i-1}}{z_{i-1}} \wedge \frac{dz_{i+1}}{z_{i+1}} \wedge \dots \wedge \frac{dz_n}{z_n}$$

for $i = 1, \dots, n$ and that such an n -tuple of $(n-1)$ -forms yield a basis in $H^{n-1}(T, \mathbb{C})$ which proves the desired statement. \square

4.3. Notation. Recall that every reductive group G is a complexification of a compact maximal subgroup K of G , i.e $G = K^{\mathbb{C}}$. Similarly if R is a closed reductive subgroup of G then $R = L^{\mathbb{C}}$ for a maximal compact subgroup L of R . Extending L to a maximal compact subgroup of G (which is unique up to conjugation by the Cartan-Iwasawa-Maltsev theorem) we can suppose that it is contained in K . We need the following Mostow Decomposition theorem (e.g., see [19], [20], and also [10][Section 3.1]).

Theorem 2. *Let K and L be as before and \mathfrak{l} (resp. $\mathfrak{l}^{\mathbb{C}}$) be the Lie algebra of K (resp. $K^{\mathbb{C}}$). Suppose that L acts on $\mathfrak{l}^{\mathbb{C}}$ by the adjoint representation. Then there exists a L -invariant linear subspace \mathfrak{m} of \mathfrak{l} such that the map $K \times_L \sqrt{-1}\mathfrak{m} \rightarrow K^{\mathbb{C}}/L^{\mathbb{C}}$ given by $(k, v) \rightarrow k \cdot \exp v L^{\mathbb{C}}$ is an isomorphism of topological K -spaces.*

4.4. Corollary. *The homogeneous space $X := G/R = K^{\mathbb{C}}/L^{\mathbb{C}}$ is homotopy equivalent to K/L and there is a natural isomorphism between $H_1(X, \mathbb{C})$ and $H^{n-1}(X, \mathbb{C})$. In particular, when G is semi-simple then $H^{n-1}(X, \mathbb{C}) = 0$ and Condition (B) holds.*

Proof. Since $X \simeq K \times_L \sqrt{-1}\mathfrak{m}$ is a vector bundle over K/L we see that K/L is a retract of X which is the first statement. Since K/L is a real n -dimensional compact manifold by the Poincare duality we have a natural isomorphism between $H_1(K/L, \mathbb{C})$ and $H^{n-1}(K/L, \mathbb{C})$ which is the second statement. In the case of a semi-simple G the fundamental group $\pi_1(K)$ is finite. Hence the exact homotopy sequence of the locally trivial fibration $K \rightarrow K/L$ implies that $\pi_1(X) \simeq \pi_1(K/L)$ is finite. Therefore $H_1(X, \mathbb{C}) = 0$ and as a result $H^{n-1}(K/L, \mathbb{C}) = 0$. Now Condition (B) holds automatically. \square

4.5. Notation. Let G be a connected linear algebraic group, R be a closed reductive subgroup of G (in particular $X = G/R$ is affine by the Matsushima theorem [17]), G_0 be a maximal reductive subgroup of G , and \mathcal{R}_u be the normal subgroup of G whose Lie algebra is the unipotent radical of the Lie algebra of G (i.e. as an affine algebraic variety \mathcal{R}_u is isomorphic to a Euclidean space). Then G_0 admits an unramified covering \hat{G}_0 which is of form $\hat{G}_0 \simeq \hat{S} \times \hat{T}$ with \hat{S} being a simply connected semi-simple (Levi) subgroup of the group \hat{G}_0 and a torus \hat{T} being the connected component of the center of \hat{G}_0 .

4.6. Lemma. *Let Notation 4.5 hold. Then $X = G/R$ can be presented as $X \simeq \hat{X}/\Gamma$ where $\hat{X} = Y \times \mathcal{R}_u$ and Γ is a finite subgroup of the center of \hat{G}_0 acting on a \hat{G}_0 -homogeneous space Y . Furthermore, there is a \hat{G}_0 -equivariant isomorphism $\varphi : Y \rightarrow$*

$X_1 \times X_2$ where $X_1 = \hat{S}/\hat{R}$ for some connected reductive subgroup \hat{R} of \hat{S} , and X_2 is a subtorus in the connected component of the center of \hat{G}_0 .

Proof. Extend R to a maximal reductive subgroup. Since such a subgroup is unique up to conjugation [18] we can suppose that R is contained in G_0 . By Mostow's theorem [18] G can be viewed as a semi-direct product of G_0 and \mathcal{R}_u . Hence as an affine algebraic variety X is isomorphic to $(G_0/R) \times \mathcal{R}_u$. Taking \hat{R}_0 as the connected identity component of the preimage of R in \hat{G}_0 we get the desired $\hat{X} = Y \times \mathcal{R}_u$ such that $Y = \hat{G}_0/\hat{R}_0$ and $X = (Y/\Gamma) \times \mathcal{R}_u = \hat{X}/\Gamma$ for some finite subgroup Γ in the center of \hat{G}_0 .

To show existence of an isomorphism φ consider the image \hat{T}_0 of $\hat{R}_0 \subset \hat{S} \times \hat{T}$ in \hat{T} under the natural projection. Let T be another subtorus of \hat{T} such that \hat{T} is naturally isomorphic to $\hat{T}_0 \times T$. Then any element $g \in \hat{G}_0$ can be presented as st_0t where $s \in \hat{S}$, $t_0 \in \hat{T}_0$, and $t \in T$. Note that for any coset $g\hat{R}_0$ one can choose a representative in the form st . Let $\hat{R} = \hat{S} \cap \hat{R}_0$ and consider the map $\varphi : \hat{G}_0/\hat{R}_0 \rightarrow \hat{S}/\hat{R} \times T$ given by $gR \rightarrow (s\hat{R}, t)$. One can see that this map is well-defined and bijective. Furthermore if we define the action of $g' = s't'_0t' \in \hat{G}_0$ on $(s\hat{R}, t)$ by formula $(s\hat{R}, t) \rightarrow (s's\hat{R}, t't)$ then φ is \hat{G}_0 -equivariant. It remains to check that \hat{R} is reductive which is a consequence of the Matsushima theorem since otherwise \hat{S}/\hat{R} (and therefore \hat{X}) is not affine. This concludes the proof. \square

4.7. Notation. Let $X = X_1 \times X_2$ and $p_i : X \rightarrow X_i$ be the natural projection. Suppose that ω_i is the volume form on X_i . Then we denote the volume form $p_1^*(\omega_1) \wedge p_2^*(\omega_2)$ by $\omega_1 \times \omega_2$. In a more general setting let τ_i be a k_i -form on X_i . Then the $(k_1 + k_2)$ -form $p_1^*(\tau_1) \wedge p_2^*(\tau_2)$ will be also denoted by $\tau_1 \times \tau_2$.

4.8. Remark. It is worth mentioning that any volume form ω on X can be presented as $\omega = \omega_1 \times \omega_2$ provided each of factors admit an algebraic volume form. Since tori and Euclidean spaces admit such forms Proposition 8.4 from Appendix implies that any algebraic volume form $\hat{\omega}$ on $\hat{X} = X_1 \times X_2 \times \mathcal{R}_u$ from Lemma 4.6 can be presented as $\omega_1 \times \omega_2 \times \omega_{\mathcal{R}_u}$ where $\omega_{\mathcal{R}_u}$ is an algebraic volume form on \mathcal{R}_u . The absence of nonconstant regular invertible functions on X_1 and \mathcal{R}_u implies that ω_1 and $\omega_{\mathcal{R}_u}$ are unique up to constant factors, i.e. $\hat{\omega}$ is determined by the choice of ω_2 .

4.9. Proposition. *Let the assumptions of Lemma 4.6 hold, ω_2 as before be an invariant volume form on the torus $X_2 \simeq T$, and $\omega_Y = \omega_1 \times \omega_2$. Then Condition (B) holds for the pair (Y, ω_Y) . Furthermore, the space $H^{n-1}(Y, \mathbb{C})$ (where n is the dimension of Y) is generated by the image of $\Theta(\text{IVF}_{\omega_Y}(Y, \Gamma))$ under the De Rham homomorphism Φ_{n-1} where $\text{IVF}_{\omega_Y}(Y, \Gamma) \subset \text{IVF}_{\omega_Y}(Y)$ consists of Γ -invariant vector fields. In particular, Condition (B) is valid for the triple (Y, ω_Y, Γ) .*

Proof. By Lemma 4.2 and Corollary 4.4 it suffices to consider the case when each of the factors X_1 and X_2 is nontrivial. Let n_i be the dimension of X_i and $n = n_1 + n_2$.

Then as we mentioned in the proof of Corollary 4.4 $H^{n_1-1}(X_1, \mathbb{C}) = 0$. Hence the Künneth formula implies that the space $H^{n-1}(X, \mathbb{C})$ is generated by the images of closed algebraic $(n-1)$ -forms that can be presented as $\omega_1 \times \tau$ where τ is a closed algebraic (n_2-1) -form on X_2 . We saw in the proof of Lemma 4.2 that such τ can be chosen in the form $\tau = \iota_\nu \omega_2$ where ν is a semi-simple vector field on the torus $T \simeq X_2$ associated with the multiplication by elements of a \mathbb{C}^* -subgroup F of T . Furthermore, we saw that such ν is of ω_2 -divergence zero. Hence $H^{n-1}(Y, \mathbb{C})$ is generated by $\Phi_{n-1}(\Theta(\text{IVF}_{\omega_Y}(Y)))$. For the last statement it remains to note that since the actions of Γ and F (generated by multiplications) commute such fields ν are Γ -invariant. \square

Though we shall not use this fact later it is worth mentioning that Condition (B) is also valid for \hat{X} from Lemma 4.6 by the following.

4.10. Lemma. *Suppose that $X = Y \times \mathbb{C}^k$ where $k \geq 1$ and $\omega = \omega_Y \times \omega_{\mathbb{C}^k}$ be an algebraic volume form on X . In the case of $k = 1$ let the image of ω_Y under the De Rham homomorphism be a generator of $H^{n-1}(Y, \mathbb{C})$ where n is the dimension of X (by Corollary 8.7 in Appendix this assumption is always true in the case when Y is a connected homogeneous space without nonconstant invertible functions). Then Condition (B) is valid for the pair (X, ω) .*

Proof. By the Künneth formula $H^{n-1}(X, \mathbb{C}) = 0$ as soon as $k \geq 2$ and Condition (B) holds. Let $k = 1$ and z be a coordinate on \mathbb{C}^k . Then by the Künneth formula ω_1 corresponds to a generator of $H^{n-1}(X, \mathbb{C}) = \mathbb{C}$. The derivative $\frac{d}{dz}$ induces a locally nilpotent derivation ξ on X which is automatically divergence-free with respect to any volume form. Note also that $\iota_\xi \omega = \omega_1$ which yields condition (B). \square

5. INCLUSION $\mathcal{B}_{n-1}(X, \Gamma) \subset \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$

To verify the assumptions of Theorem 1, besides Condition (B) one needs to show that $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains $\mathcal{B}_{n-1}(X, \Gamma)$ as in Proposition 3.6, e.g., to check strong Condition (A). For homogeneous spaces of semi-simple groups this follows from the next lemma and Proposition 2.8.

5.1. Lemma. *Let X_1 be a homogeneous space of a semi-simple group \hat{S} , Γ_1 be a finite subgroup of the center of \hat{S} , and v_1, v_2 be non-collinear vectors in the space $T_{x_1}X_1$ at some point $x_1 \in X_1$. Suppose that \mathcal{N}_0 is the set of locally nilpotent vector fields on X_1 associated with multiplications by \mathbb{C}_+ -subgroups of \hat{S} and that H is the group of algebraic automorphisms of X generated by elements of phase flows associated with elements from the set \mathcal{N} of all locally nilpotent vector fields of form $f\xi$ where $\xi \in \mathcal{N}_0$ and the function $f \in \text{Ker } \xi$ is Γ_1 -invariant.*

Then the orbit O of $v_1 \wedge v_2$ under the action of the isotropy group H_{x_1} generates the whole wedge-product space $V = \Lambda^2 T_{x_1}X_1$.

Proof. Let ν be a completely integrable algebraic vector field and $f \in \text{Ker } \nu$ be a function such that $f(x_1) = 0$. Then the phase flow φ_t associated with $f\nu$ generates an

isomorphism $T_{x_1}X_1 \rightarrow T_{x_1}X_1$ given by the formula

$$(3) \quad w \rightarrow w + \mathrm{d}f(w)v$$

where $v = \nu(x_1)$. Note that each $\xi \in \mathcal{N}_0$ corresponds a nilpotent element of the Lie algebra of \hat{S} . Hence the set $\{\xi(x)|\xi \in \mathcal{N}_0\}$ generates T_xX_1 for every $x \in X_1$ which implies that X_1 is an H -flexible variety in terminology of [3]. That is, H acts transitively on X_1 . In particular replacing if necessary x_1 by $h(x_1)$ for some $h \in H$ we can treat x_1 as a general point of X_1 . Assume that $\nu = h_*(\nu_0)$ for some $\nu_0 \in \mathcal{N}_0$. Since it is enough to consider the case of $\dim X_1 \geq 3$, we can suppose that v_2 is not a linear combination of v_1 and $v = \nu(x_1)$. This means that if $\rho : X_1 \rightarrow Q = \mathrm{Spec} \mathrm{Ker} \nu$ is the quotient morphism then $u_1 = \rho_*(v_1)$ and $u_2 = \rho_*(v_2)$ are not collinear, i.e. for a general regular function f on Q one has $\mathrm{d}f(u_1) \neq 0$ and $\mathrm{d}f(u_2) = 0$. Furthermore, by construction the H -action on X_1 commutes with the Γ_1 -action which yields a Γ_1 -action on Q . Thus we can choose f to be Γ_1 -invariant, i.e. $f\nu \in \mathcal{N}$. Treating f as a function on X_1 we get $\mathrm{d}f(v_1) \neq 0$ and $\mathrm{d}f(v_2) = 0$. In combination with the formula (3) this implies that the span of O contains the wedge product $v_1 \wedge v$. Now choose another locally nilpotent derivation for which the value u at x_1 is not a linear combination of v_1 and v . Repeating the argument as before we see that the space of O contains $u \wedge v$. By [3][Corollary 4.3] u and v can be chosen sufficiently general (more precisely, the set $\{h_*(\nu)(x_1)|\nu \in \mathcal{N}_0, h \in H\}$ coincides with $T_{x_1}X_1$). Hence we get the desired conclusion. \square

5.2. Remark. Let $\mathrm{SAut}(X)$ be the subgroup of the group $\mathrm{Aut}(X)$ of algebraic automorphisms of X generated by elements of all algebraic one-parameter unipotent subgroups of $\mathrm{Aut}(X)$. Recall that according to one of equivalent definitions [3] a normal affine algebraic variety X is flexible⁴ if $\mathrm{SAut}(X)$ acts transitively on the smooth part of X . In this terminology one can have the following straightforward extension of Lemma 5.1: for every smooth point $x \in X$ the isotropy group $(\mathrm{SAut}(X))_x$ induces an irreducible action on $\Lambda^2 TX$. In the presence of a semi-compatible pair this yields a strong Condition (A). Thus by Theorem 1 we have the following fact which will not be used later.

Theorem 3. *Let X be a smooth flexible variety equipped with an algebraic volume form ω such that $H^{n-1}(X, \mathbb{C}) = 0$ where $n = \dim X$. Suppose that X admits a semi-compatible pair of divergence-free vector fields. Then X has AVDP.*

Taking into consideration Example 2.3 we have the following.

5.3. Corollary. *Let X be a smooth flexible variety equipped with an algebraic volume form ω such that $H^{n-1}(X, \mathbb{C}) = 0$ where $n = \dim X$. Suppose that either X admits*

⁴The class of flexible varieties includes homogeneous spaces of extensions of semi-simple groups by unipotent radicals, non-degenerate toric varieties, cones over flag varieties and del Pezzo surfaces (of degree at least 4), hypersurfaces of form $\{uv = p(\bar{x})\} \subset \mathbb{C}_{u,v,\bar{x}}^{n+2}$, homogeneous Gizatullin surfaces (except for $\mathbb{C}^* \times \mathbb{C}$), etc..

a fixed point free non-degenerate algebraic SL_2 -action or $X = X_1 \times X_2$ where X_i is a flexible variety of dimension at least 1 equipped with a volume form ω_i such that $\omega = \omega_1 \times \omega_2$. Then X has AVDP.

For an effective application of Proposition 3.6 we need also methods of constructing $\xi \in \text{Lie}_{\text{alg}}^{\hat{\omega}}(\hat{X})$ for which $\iota_{\xi}\hat{\omega}$ is an exact form.

5.4. Proposition. *Let X be a Γ -variety equipped with a volume form ω and $\xi \in \text{Lie}_{\text{alg}}^{\omega}(X, \Gamma)$, $\eta \in \text{IVF}_{\omega}(X, \Gamma)$ be such that $[\xi, \eta] = 0$ and $\text{Ker } \eta \neq \text{Ker } \xi$. Then for some $f \in \text{Ker } \eta \setminus \text{Ker } \xi$ the field $\nu = \xi(f)\eta$ belongs to $\text{Lie}_{\text{alg}}^{\omega}(X, \Gamma)$ and the form $\iota_{\nu}\omega$ is exact.*

Proof. Let $\bar{f} = \{f_1, \dots, f_m\}$ be the Γ -orbit of f . Taking symmetric polynomials $\sigma_1(\bar{f}), \dots, \sigma_m(\bar{f})$ we see that at least one of them is nonzero and contained in $\text{Ker } \eta \setminus \text{Ker } \xi$ (because $\text{Ker } \xi$ is integrally closed in $\mathbb{C}[X]$). Thus f can be always chosen Γ -invariant. Then $f\eta \in \text{IVF}_{\omega}(X, \Gamma)$ and $[\xi, f\eta] = \xi(f)\eta$ which implies the desired conclusion by Proposition 4.1. \square

As a consequence we get a result first proven by Varolin [22].

5.5. Corollary. *Any torus $X = (\mathbb{C}^*)^n$ has AVDP with respect to an invariant volume form ω .*

Proof. Recall that Condition (B) from Theorem 1 is valid for X by Lemma 4.2. Hence aside from a trivial case of $k = 1$ it suffices to check strong Condition (A) from Proposition 3.6. For $i \neq j$ the semi-simple vector fields ν_i and ν_j commute where $\nu_i = z_i \partial / \partial z_i$. Hence for any nonconstant Laurent polynomial f in z_j Proposition 5.4 implies that $\nu'_i = \nu_j(f)\nu_i \in \text{Lie}_{\text{alg}}^{\omega}(X)$ corresponds to an exact form. Consideration of wedge products $\nu'_i \wedge \nu'_j$ implies strong Condition (A) and the desired conclusion \square

5.6. Notation. In the rest of this section we consider a situation when X is isomorphic to the direct product $X_1 \times X_2$ of n_i -dimensional Γ_i -varieties X_i equipped with volume forms ω_i . We equip X with the volume form $\omega = \omega_1 \times \omega_2$ and treat it as an n -dimensional Γ -variety where Γ is a subgroup of $\Gamma_1 \times \Gamma_2$ acting naturally on X . For each point $\bar{x} = (x_1, x_2) \in X$ we have the natural embedding $T_{x_i}X_i \hookrightarrow T_{\bar{x}}X$ such that $T_{\bar{x}}X \simeq T_{x_1}X_1 \oplus T_{x_2}X_2$ which enables us to treat $T_{x_i}X_i$ (resp. $\Lambda^k T_{x_i}X_i$) as a subspace of $T_{\bar{x}}X$ (resp. $\Lambda^k T_{\bar{x}}X$). We let $V_i = \Lambda^2 T_{x_i}X_i \subset \Lambda^2 T_{\bar{x}}X$ and denote by V the smallest subbundle of $\Lambda^2 TX$ that contains all wedge-products of form $v_1 \wedge v_2$ where $v_i \in T_{x_i}X_i \subset T_{\bar{x}}X$ with $\bar{x} = (x_1, x_2)$ running over X . Furthermore, for every vector field ξ on X_i the embedding $T_{x_i}X_i \hookrightarrow T_{\bar{x}}X$ yields an induced vector field on X which will be denoted by ξ' . We treat also $\mathbb{C}[X_i]$ as a natural subring of $\mathbb{C}[X]$.

5.7. Proposition. *Let Notation 5.6 hold, each X_i has Γ_i -AVDP with respect to ω_i , and let $\Gamma = \Gamma_1 \times \Gamma_2$. Then X has Γ -AVDP with respect to ω .*

In [14][Proposition 4.3] this fact was proven for trivial Γ_1 and Γ_2 but the proof remains valid in the general case with straightforward adjustments (more precisely,

instead of the rings of regular function $\mathbb{C}[X_i]$ one need to consider the subrings of Γ_i -invariant functions $\mathbb{C}[X_i, \Gamma_i]$. Hence we omit it and concentrate on a more difficult case of $\Gamma \neq \Gamma_1 \times \Gamma_2$.

Note that if $\xi_i \in \text{IVF}_{\omega_i}(X_i, \Gamma_i)$ then $[\xi'_1, \xi'_2] = 0$. Hence by Proposition 5.4 we have the following.

5.8. Lemma. *Let Notation 5.6 hold and ξ_i be as above. Then for every $f \notin \text{Ker } \xi_1 \subset \mathbb{C}[X_1]$ the field $\xi''_2 = \xi_1(f)\xi'_2 \in \text{Lie}_{\text{alg}}^\omega(X)$ corresponds to an exact form under isomorphism $\Theta : \text{AVF}_\omega(X) \rightarrow \mathcal{C}_{n-1}(X)$. Furthermore, if f is Γ_1 -invariant (which can be always found as in Proposition 5.4) then $\xi''_2 \in \text{Lie}_{\text{alg}}^\omega(X, \Gamma)$.*

5.9. Lemma. *Suppose that Notation 5.6 holds and $\{\xi_{ij}\}_{j=1}^k \subset \text{IVF}_{\omega_i}(X_i, \Gamma_i)$, $i = 1, 2$ is such that the set $\{\xi_{ij}(x_i)\}_{j=1}^k$ generates $T_{x_i}X_i$ at every point $x_i \in X_i$. Then $\mathcal{B}_{n-1}(X, \Gamma) \setminus \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ is contained in $\mathcal{V}_1 + \mathcal{V}_2$ where \mathcal{V}_1 (resp. \mathcal{V}_2) is the subspace of $\mathcal{B}_{n-1}(X, \Gamma)$ that is the $\mathbb{C}[X_2]$ -module (resp. $\mathbb{C}[X_1]$ -module) generated by forms $\tau_1 \times \omega_2$ (resp. $\omega_1 \times \tau_2$) with τ_i being an exact $(n_i - 1)$ -form on X_i .*

Proof. By Lemma 5.8 for every Γ_2 -invariant $f \notin \text{Ker } \xi_{2j} \subset \mathbb{C}[X_2]$ the field $\xi''_{1j} = \xi_{2l}(f)\xi'_{1j} \in \text{Lie}_{\text{alg}}^\omega(X, \omega)$ corresponds to an exact form under isomorphism Θ . Since the set $\{\xi_{ij}(x_i)\}_{j=1}^k$ generates $T_{x_i}X_i$ at every point $x_i \in X_i$ with an appropriate choice of l and f we can suppose that ξ''_{1j} does not vanish at any given point $\bar{x} = (x_1, x_2) \in X$ provided $\xi_{1j} \neq 0$ at x_1 . This implies that elements $\{\xi''_{1j}(\bar{x}) \wedge \xi''_{2l}(\bar{x})\}_{i,l}$ generate the fiber $V_{\bar{x}}$ of V (from Notation 5.6) at every $\bar{x} \in X$ where ξ''_{2l} has the meaning similar to ξ''_{1j} .

By Example 2.3(1) the fields ξ''_{1j} and ξ''_{2l} are Γ -semi-compatible and the associate Γ -ideal I_{jl} is $\mathbb{C}[X]$. That is, elements $\{I_{jl}(\bar{x})\xi''_{1j}(\bar{x}) \wedge \xi''_{2l}(\bar{x})\}_{i,l}$ generate $V_{\bar{x}}$. By Proposition 3.6 $D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains $\mathcal{C}_{n-2}^V(X, \Gamma)$ (from Notation 3.5), i.e. $\mathcal{C}_{n-2}(X, \Gamma) \setminus D^{-1} \circ \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma)) \subset \mathcal{C}_{n-2}^{V_1}(X, \Gamma) + \mathcal{C}_{n-2}^{V_2}(X, \Gamma)$. Furthermore, note that $D(\mathcal{C}_{n-2}^{V_i}(X, \Gamma))$ is a $\mathbb{C}[X_j]$ -module for $j \neq i$ which admits a filtration by finitely generated submodules and contains \mathcal{V}_i as a submodule. The restriction of say both $D(\mathcal{C}_{n-2}^{V_1}(X, \Gamma))$ and \mathcal{V}_1 to every fiber $X_1 \times x_2 \simeq X_1$ coincides with $\mathcal{B}_{n_1-1}(X_1, \Gamma_1) \times \omega_2$. Hence by Lemma 2.9 $D(\mathcal{C}_{n-2}^{V_i}(X, \Gamma)) = \mathcal{V}_i$.

□

5.10. Lemma. *Let the Notation 5.6 hold and let $\{\xi_{ij}, \eta_{ij}\}_{j=1}^k$ be a collection of semi-compatible pairs from $\text{IVF}_{\omega_i}(X_i, \Gamma_i)$ satisfying strong Condition (A) for $i = 1, 2$. Let also a group F of algebraic automorphisms act transitively on X so that this action commutes with the Γ -action and up to constant factors preserves ω . Then $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ contains $\mathcal{B}_{n-1}(X, \Gamma)$.*

Proof. Let us consider vector fields ξ''_{ij} (resp. η''_{ij}) related to ξ_{ij} (resp. η_{ij}) as ξ''_2 to ξ_2 in Lemma 5.8, i.e. each $(\xi''_{ij}, \eta''_{ij})$ is still a Γ -semi-compatible pair with a non-trivial associate Γ -ideal I_{jl} . Then for a general point $\bar{x} = (x_1, x_2) \in X$ the set $\{I_{jl}(\bar{x})\xi''_{1j}(\bar{x}) \wedge \eta''_{1j}(\bar{x})\}_{j=1}^k$ generates the image of the subspace $\Lambda^2 T_{x_1}X_1$ in $\Lambda^2 T_{\bar{x}}X$ under the natural embedding. Similarly one can take care of $\Lambda^2 T_{x_2}X_2 \subset \Lambda^2 T_{\bar{x}}X$. Enlarging $\{\xi_{ij}, \eta_{ij}\}_{j=1}^k$ one can suppose that $\{\xi_{ij}\}_{j=1}^k$ satisfies the assumption of Lemma 5.9.

Hence the associate Γ -ideal of (ξ''_{1j}, ξ''_{2l}) is $\mathbb{C}[X, \Gamma]$ and the fiber $V_{\bar{x}}$ of V is generated by elements of form $\xi''_{1j}(\bar{x}) \wedge \xi''_{2j}(\bar{x})$. Since $\Lambda^2 T_{x_1} X_1 + \Lambda^2 T_{x_2} X_2 + V_{\bar{x}} = \Lambda^2 T_{\bar{x}} X$ we are under the assumptions of Corollary 3.7 which implies the inclusion $\Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma)) \supset \mathcal{B}_{n-1}(X, \Gamma)$. \square

5.11. Remark. Note that Lemma 5.10 is valid even if one of factors, say, X_2 is one-dimensional. That is, one can consider X_2 equal to \mathbb{C}_z (resp. \mathbb{C}^*) with $\omega_2 = dz$ (resp. $\omega_2 = \frac{dz}{z}$).

6. MAIN THEOREM

The aim of this section is the following.

Theorem 4. *Let Notation 4.5 hold, i.e. R is a closed reductive subgroup of a linear algebraic group G , $X = G/R$ is the homogeneous space of left cosets which by Lemma 4.6 can be presented as $X \simeq (Y/\Gamma) \times \mathcal{R}_u$ where \mathcal{R}_u is isomorphic to a Euclidean space and Y is a homogeneous space of a reductive group \hat{G}_0 that is an unramified covering of a maximal reductive subgroup G_0 of G and has the following properties*

- (i) *a Levi semi-simple subgroup \hat{S} of \hat{G}_0 is simply connected;*
- (ii) *$Y = X_1 \times X_2$ where $X_1 = \hat{S}/\hat{R}$ for some connected reductive subgroup \hat{R} of \hat{S} , and X_2 is a subtorus in the connected component of the center of \hat{G}_0 ;*
- (iii) *Γ is a finite subgroup of the center of \hat{G}_0 .*

Let $\hat{\omega} = \omega_1 \times \omega_2 \times \omega_{\mathcal{R}_u}$ be an algebraic volume form on $\hat{X} = X_1 \times X_2 \times \mathcal{R}_u$ as in Remark 4.8. Suppose that ω_2 is an invariant volume form on the torus X_2 .

Then \hat{X} has Γ -AVDP (with respect to $\hat{\omega}$).

Before presenting the proof let us consider one case related to two-dimensional homogeneous spaces where the argument is very specific since Proposition 2.8 is not valid and therefore the technique of semi-compatible fields does not work.

6.1. Example. Consider $G = SL_2$ as a subvariety of $\mathbb{C}_{a_1, a_2, b_1, b_2}^4$ given by $a_1 b_2 - a_2 b_1 = 1$, i.e. matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

are elements of G . Let $T \simeq \mathbb{C}^*$ be the torus consisting of the diagonal elements and N be the normalizer of T in SL_2 . That is, $N/T \simeq \mathbb{Z}_2 = \Gamma_1$ where the matrix

$$A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in N$$

generates the nontrivial coset of N/T .

By Proposition 8.4 the homogeneous space $X_1 = G/T$ possess a G -invariant volume form ω_1 while $G/N \simeq X_1/\Gamma_1$ does not (see Example 8.5)). It is worth mentioning that X has AVDP since it is isomorphic to the hypersurface $uv = x^2 - 1$ in $\mathbb{C}_{u, v, x}^3$ and

such hypersurfaces were dealt with in [14]. Also the action of Γ_1 on X_1 is given by $(u, v, x) \rightarrow (-u, -v, -x)$ ⁵.

Furthermore Γ_1 -AVDP for X_1 was established in [16]. As an application of Lemma 5.9 let us show \mathbb{Z}_2 -AVDP for a more complicated object: $X = X_1 \times X_2$ where $X_2 = \mathbb{C}_z^*$ and the action of $\Gamma \simeq \mathbb{Z}_2$ is given by $(u, v, x, z) \rightarrow (-u, -v, -x, -z)$. That is, we have the group $F = SL_2 \times \mathbb{C}^*$ acting naturally on X so that the action commutes with Γ -action and preserves the volume form $\omega = \omega_1 \times \omega_2$ (where ω_2 is an invariant volume form on the torus X_2).

Let $\Gamma_2 \simeq \mathbb{Z}_2$ act on $X_2 \simeq \mathbb{C}_z^*$ by $z \rightarrow -z$. Since for $i = 1, 2$ the variety X_i has Γ_i -AVDP there are functions $\{\xi_{ij}\}_{j=1}^k$ satisfying the assumptions of Lemma 5.9. Thus in order to establish $\mathcal{B}_{n-1}(X, \Gamma) \subset \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$ one needs to check that $\Theta(\text{Lie}_{\text{alg}}^\omega(\hat{X}, \Gamma))$ contains subspaces \mathcal{V}_1 and \mathcal{V}_2 from Lemma 5.9. Note also that $\mathcal{V}_1 = \mathcal{V}_1^+ + \mathcal{V}_1^-$ where \mathcal{V}_1^+ (resp. \mathcal{V}_1^-) consists of forms $f\tau_1 \times \omega_2$ with $\tau_1 \in \mathcal{B}_{n_1-1}(X_1)$ being Γ_1 -invariant (resp. Γ_1 -anti-invariant) and $f \in \mathbb{C}[X_2]$ being Γ_2 -invariant (resp. Γ_2 -anti-invariant). By [16][Theorem 3.6 and Remark 3.7] such an anti-invariant (resp. invariant) τ_1 belongs to the span of $\Theta(\text{IVF}_{\omega_1}^-(X_1, \Gamma_1))$ (resp. $\Theta(\text{IVF}_{\omega_1}^+(X_1, \Gamma_1))$ where $\text{IVF}_{\omega_1}^-(X_1, \Gamma_1) \subset \text{IVF}_{\omega_1}(X_1)$ is the subset of anti-invariant fields) and therefore in both cases $f\tau_1 \times \omega_2 \in \Theta(\text{Lie}_{\text{alg}}^\omega(\hat{X}, \Gamma))$ (since for $\xi \in \text{IVF}_{\omega_1}^-(X_1, \Gamma_1)$ the field $z\xi'$ is Γ -invariant on X). Note that by the same argument the similar space $\mathcal{V}_2^+ \subset \Theta(\text{Lie}_{\text{alg}}^\omega(\hat{X}, \Gamma))$ while \mathcal{V}_2^- is trivial since there are no nonzero \mathbb{Z}_2 -anti-invariant divergence-free forms on \mathbb{C}^* which concludes the proof of the inclusion $\mathcal{B}_{n-1}(X, \Gamma) \subset \Theta(\text{Lie}_{\text{alg}}^\omega(X, \Gamma))$. Condition (B) is also valid since for the vector field $\nu = z\partial/\partial z$ the form $\iota_\nu\omega$ is a generator of $H^2(X, \mathbb{C}) \simeq \mathbb{C}$. Hence Theorem 1 yields the desired conclusion.

6.2. Proof of Theorem 4. Since \mathcal{R}_u is a Euclidean space which always has AVDP (with respect to any volume form since such a form is unique up to a constant factor) we can suppose that the unipotent radical \mathcal{R}_u is trivial because of Proposition 5.7.

Thus from now on G is reductive. That is, $X = \hat{X}/\Gamma$ where $\hat{X} = X_1 \times X_2 = \hat{G}_0/\Gamma$ and X_1 can be assumed nontrivial by virtue of Lemma 4.2. Since Γ is a subgroup of the center of \hat{G}_0 its action commutes with the actions of \mathbb{C}_+ and \mathbb{C}^* -subgroups of \hat{G}_0 induced by the left multiplication. In particular, the semi-simple vector fields ν_i that appeared in the proof of Corollary 5.5 are Γ -invariant. One can suppose that each ν_i is tangent to X_2 . Since ω_2 is an invariant volume form on X_2 these vector fields are of zero divergence with respect to this form. Thus we have a collection of divergence-free Γ -invariant vector fields $\{\xi_{2j}, \eta_{2j}\}_{j=1}^k$ on X_2 such that $\{\xi_{2j}(x_2) \wedge \eta_{2j}(x_2)\}_{j=1}^k$ generates the whole wedge-product space $\Lambda^2 T_{x_2} X_2$ at any point $x_2 \in X_2$ as required in Lemma 5.10.

⁵Indeed, ring of T -invariant regular functions on G is generated by $u = a_1 b_1, v = a_2 b_2, y = a_1 b_2$, and $z = a_2 b_1$ where $y = z + 1$. Hence X_1 is isomorphic to the hypersurface $uv = z(z+1)$ in $\mathbb{C}_{u,v,z}^3$. Let $x = z + 1/2$. Then X is isomorphic to $uv = x^2 - 1/4$ in $\mathbb{C}_{u,v,x}^3$ and replacing (u, v, x) by $(2u, 2v, 2x)$ we get the desired equation. The formula for the \mathbb{Z}_2 -action (induced by multiplication by A_0) is now a straightforward computation.

By Propositions 2.5 and 2.8 there is a semi-compatible pair (ξ, η) of locally nilpotent vector fields on X_1 whenever $\dim X_1 \geq 3$. Let v_1 and $v_2 \in T_{x_1}X_1$ be the values of these vector fields at some general point $x_1 \in X_1$ (in particular these vectors are not collinear). Let Γ_1 be the image of Γ under the natural projection $\hat{G}_0 = \hat{S} \times \hat{T} \rightarrow \hat{S}$ (i.e. Γ_1 is a finite subgroup of the center of \hat{S}). Suppose that H is the group of automorphisms of X_1 described in Lemma 5.1 (in particular, being generated by elements of \mathbb{C}_+ -actions H preserves any algebraic volume form). Then the H_{x_1} -orbit of $v_1 \wedge v_2$ generates $\Lambda^2 T_{x_1}X_1$. Thus we get a collection of divergence-free vector fields $\{\xi_{1j}, \eta_{1j}\}_{j=1}^k$ on X_1 such that $\{\xi_{1j}(x_1) \wedge \eta_{1j}(x_1)\}_{j=1}^k$ generates the whole wedge-product space $\Lambda^2 T_{x_1}X_1$ at $x_1 \in X_1$. Because of transitivity of the H -action one can suppose that this is true for any point in X_1 . Let the fields ξ'_{ij}, η'_{ij} on $X_1 \times X_2$ have the same meaning as in Notation 5.6. By construction they are Γ -invariant and one can suppose that $\{\xi'_{ij}\}_{j=1}^k$ generate $T_{x_i}X_i$ at every $x_i \in X_i$. Hence Lemma 5.10 implies that $\Theta(\text{Lie}_{\text{alg}}^\omega(\hat{X}, \Gamma))$ contains $\mathcal{B}_{n-1}(\hat{X}, \Gamma)$. By Proposition 4.9 Condition (B) holds for \hat{X} . Hence Theorem 1 yields the desired conclusion when $\dim X_1 \geq 3$.

In the case of $\dim X_1 = 2$ choose a subgroup $\hat{S}' \simeq SL_2$ of \hat{S} so that it is not contained in \hat{R} . Furthermore, by the Cartan-Iwasawa-Maltsev theorem we can organize this choice so that maximal compact subgroups $\hat{S}'_{\mathbb{R}}$ and $\hat{R}_{\mathbb{R}}$ of \hat{S}' and R respectively are contained in the same maximal compact subgroup $\hat{S}_{\mathbb{R}}$ of \hat{S} . Note that the group $\hat{S}'_{\mathbb{R}} \cap \hat{R}_{\mathbb{R}}$ is of dimension 1 since otherwise the complexification of $\hat{S}'_{\mathbb{R}} \cap \hat{R}_{\mathbb{R}}$ is a two-dimensional reductive subgroup of \hat{S}' but the only two-dimensional reductive group $\mathbb{C}^* \times \mathbb{C}^*$ is not contained in SL_2 . Hence the image of $\hat{S}'_{\mathbb{R}}$ under the quotient morphism $\hat{S}_{\mathbb{R}} \rightarrow \hat{S}_{\mathbb{R}}/\hat{R}_{\mathbb{R}}$ is surjective and therefore the image of \hat{S}' under the quotient morphism $\hat{S} \rightarrow \hat{S}/\hat{R}$ is also surjective. Thus we can suppose that $\hat{S} = SL_2$. Since maximal tori are the only proper connected reductive subgroups of SL_2 we have $\hat{R} \simeq \mathbb{C}^*$ and $X_1 \simeq SL_2/\mathbb{C}^*$.

Hence X_1 possesses an algebraic volume form (see, Proposition 8.4) and also AVDP [16]. Let Γ_1 be as before (i.e. Γ_1 is at most \mathbb{Z}_2 since it is a subgroup of the center of SL_2) and let Γ_2 be the image of Γ under the natural projection $\hat{G}_0 = \hat{S} \times \hat{T} \rightarrow \hat{T}$. Then Γ may be viewed as a subgroup of $\Gamma_1 \times \Gamma_2$. If they coincide then we are done by Proposition 5.7 and the result of [16] about \mathbb{Z}_2 -AVDP for X_1 . If not then $\Gamma = \Gamma'_1 \oplus \Gamma'_2$ where $\Gamma'_1 = \mathbb{Z}_2$ and Γ'_2 is the kernel of the natural homomorphism $\hat{G}_0 \rightarrow \hat{S}$. Hence replacing the torus $X_2 = T$ by T/Γ'_2 we can suppose that $\Gamma = \mathbb{Z}_2$. Furthermore, one can present T now as $T = T_1 \times T_2$ where $T_1 \simeq \mathbb{C}^*$ contains the generator of $\Gamma_2 = \mathbb{Z}_2$ and T_2 is another torus. Hence X is the product of $(X_1 \times T_1)/\Gamma$ and T_2 and by virtue of Proposition 5.7 we can suppose now that $X_2 = T_1$. Now the desired conclusion follows from Example 6.1. \square

6.3. Remark. It is worth mentioning that we do not assume existence of a volume form on $X = \hat{X}/\Gamma$ in Theorem 1.

6.4. Corollary. *Let a homogeneous space $X = G/R$ where G is a linear algebraic group and R is a closed reductive subgroup of G . Suppose that X has a G -invariant algebraic volume form ω . Then X has AVDP with respect to ω .*

Proof. Let $\hat{X} = X_1 \times X_2 \times \mathcal{R}_u$ be from Theorem 4 and let $\hat{\omega} = \omega_1 \times \omega_2 \times \omega_{\mathcal{R}_u}$ be an algebraic volume form on \hat{X} as in Notation 4.8. Recall that \hat{G}_0 acts naturally on \hat{X} so that the action of $t \in T \simeq X_2 \subset \hat{G}_0$ on $x = (x_1, x_2, r) \in \hat{X}$ is given by $t.x = (x_1, tx_2, r)$ (see the proof of Lemma 4.6). In particular, $\hat{\omega}$ is T -invariant iff ω_2 is. When $\hat{\omega}$ is induced by ω it must be T -invariant. Hence the assumptions of Theorem 1 hold, i.e. $\text{Lie}_{\text{alg}}^{\hat{\omega}}(\hat{X}, \Gamma) = \text{AVF}_{\hat{\omega}}(\hat{X}, \Gamma)$. Now the natural isomorphisms $\text{Lie}_{\text{alg}}^{\hat{\omega}}(\hat{X}, \Gamma) \simeq \text{Lie}_{\text{alg}}^{\omega}(X)$ and $\text{AVF}_{\hat{\omega}}(\hat{X}, \Gamma) \simeq \text{AVF}_{\omega}(X)$ imply the desired conclusion. \square

7. SURFACES $p(x) + q(y) + xyz = 1$

Theorem 4 is not the only application of the basic idea behind our criterion. In the case of a smooth affine simply connected surface S equipped with an algebraic volume form ω Condition (B) is trivial since $H^1(S, \mathbb{C}) = 0$. Furthermore, by the Grothendieck theorem [9] for every $f \in \mathbb{C}[S]$ there is $\xi \in \text{AVF}_{\omega}(S)$ for which $\iota_{\xi}\omega = \text{d}f$ and the equality $\Theta(\text{Lie}_{\text{alg}}^{\omega}(S)) = \mathcal{B}_1(S)$ (which implies $\text{Lie}_{\text{alg}}^{\omega}(S) = \text{AVF}_{\omega}(S)$) becomes equivalent to the fact that such a ξ can be chosen in $\text{Lie}_{\text{alg}}^{\omega}(S)$. The next technical fact is useful for verification of this condition.

7.1. Proposition. *Let S be a smooth affine surface equipped with an algebraic volume form ω and $\xi \in \text{AVF}_{\omega}(S)$ be nonzero. Suppose that f is a regular function such that $\iota_{\xi}\omega = \text{d}f$. Then $L_{\xi}(f) = 0$ ⁶. Furthermore, suppose that S is rational, there are no nonconstant invertible functions on S , and ξ does not vanish identically on any divisor in S . Then the kernel of ξ in $\mathbb{C}[S]$ coincides with $\text{Ker } \xi = \mathbb{C}[f]$.*

Proof. By Formula (2) one has

$$L_{\xi}(f) = \iota_{\xi}\text{d}f + \text{d}\iota_{\xi}f = \iota_{\xi}\text{d}f = \iota_{\xi}\iota_{\xi}\omega = 0$$

which yields the first statement. For the second statement note that $\text{Ker } \xi$ is of transcendence degree 1 over \mathbb{C} . Indeed, $\text{Ker } \xi \neq \mathbb{C}$ since f is not constant and $\text{Ker } \xi$ cannot be of transcendence degree 2 since otherwise being algebraically closed in $\mathbb{C}[S]$ it coincides with $\mathbb{C}[S]$. Thus for any $g \in \text{Ker } \xi$ the image of the map $(f, g) : S \rightarrow \mathbb{C}^2$ is a curve C . Since S is rational C is rational. Furthermore C does not admit nonconstant invertible functions. Hence C is a polynomial curve, i.e. the ring of regular functions on its normalization is isomorphic to $\mathbb{C}[h]$ where h is a rational continuous function on C . In particular h generates a continuous rational function on S (denoted by the same symbol h) which is regular because of the smoothness of S . Suppose that $f = p(h)$ where p is a polynomial of degree at least 2. Then $\iota_{\xi}\omega = \text{d}f = p'(h)\text{d}h$ where the left-hand side does not vanish on any divisor of S while the right-hand side has a nontrivial zero locus $p'(h) = 0$ which is a divisor. This contradiction concludes the proof. \square

7.2. Notation. We consider a hypersurface S in $\mathbb{C}_{x,y,z}^3$ given by an equation

$$p(x) + q(y) + xyz = 1, \text{ i.e. } z = \frac{1 - p(x) - q(y)}{xy}$$

⁶This statement remains valid when S is a complex surface, and ω, ξ , and f are holomorphic.

where p and q are polynomials such that $p(0) = q(0) = 0$ and $1 - p(x)$ and $1 - q(y)$ have simple roots only. Note that S contains the torus $T \simeq \mathbb{C}_x^* \times \mathbb{C}_y^*$ and up to a constant factor

$$\omega = \frac{dx \wedge dy}{xy}$$

is the only algebraic volume form on T that extends regularly to S .

7.3. Remark. One can check that S is obtained via half-locus attachments (in terminology of Fujita [8]) to the boundary of T at points $(x_1, 0), \dots, (x_k, 0)$ and $(0, y_1), \dots, (0, y_l)$ where x_1, \dots, x_k (resp. y_1, \dots, y_l) are the roots of $1 - p$ (resp. $1 - q$). In particular S is simply connected, has no nonconstant invertible functions, and of logarithmic Kodaira dimension 0. The last fact implies that S does not admit nontrivial algebraic \mathbb{C}_+ -actions and it can be also shown that it does not have nontrivial algebraic \mathbb{C}^* -actions either. Nevertheless S is transitive with respect to the group of holomorphic automorphisms generated by elements of phase flows of completely integrable algebraic vector fields (it is enough to use the algebraic vector fields listed in Lemma 7.4 below). Hence it is interesting to find out whether it has ADP or AVDP (with respect to ω). For the special case of $p(x) = x$ and $q(y) = y$ we described in [13] all completely integrable algebraic vector fields on S which turned out to be divergence-free, i.e. ADP does not hold. However as we see below AVDP is valid even in the general case.

7.4. Lemma. *Every regular function f on S can uniquely be written in the form*

$$(4) \quad f = a_0 + \sum_{i=1}^N a_i x^i + \sum_{i=1}^N b_i y^i + \sum_{i=1}^N c_i z^i + \sum_{i,j=1}^N a_{ij} x^i y^j + \sum_{i,j=1}^N b_{ij} x^i z^j + \sum_{i,j=1}^N c_{ij} y^i z^j$$

and the vector fields

$$\delta_z = (q'(y) + xz)\partial/\partial x - (p'(x) + yz)\partial/\partial y,$$

$$\delta_y = -xy\partial/\partial x + ((p'(x) + yz)\partial/\partial z,$$

$$\text{and } \delta_x = -xy\partial/\partial y + (q'(y) + xz)\partial/\partial z$$

are completely integrable on S and of ω -divergence zero. Furthermore, $\delta_z, \delta_y, \delta_x$ vanish on a finite set only and their kernels are $\mathbb{C}[z], \mathbb{C}[y],$ and $\mathbb{C}[x]$ respectively.

Proof. The first statement is a consequence of the equation $p(x) + q(y) + xyz = 1$. For δ_x and δ_y all claims follow from the fact that these fields are the images of the fields $-xy\partial/\partial y$ and $-xy\partial/\partial x$ on T under the natural embedding $T \rightarrow S$. For δ_z it is a straightforward computation and we shall check only the fact that $\text{Ker } \delta_z = \mathbb{C}[z]$. Since the $\mathbb{C}[z]$ is contained in the kernel by Proposition 7.1 it suffices to show that z cannot be presented as a nonlinear polynomial $r(h)$ of another function h on S . This follows immediately from the fact that the differential of z vanishes at the set given by $p'(x)x + (1 - p(x) - q(y)) = q'(y)y + (1 - p(x) - q(y)) = 0$ which is finite. Hence we are done.

□

Theorem 5. *The surface S from Notation 7.2 has ADVP.*

Proof. As we mentioned before for every $f \in \mathbb{C}[S]$ there is $\xi \in \text{AVF}_\omega(S)$ for which $\iota_\xi \omega = df$. Let \mathcal{V} be the subspace of $\mathbb{C}[S]$ consisting of all functions f with $a_0 = 0$ in Formula (4). Then the map $f \rightarrow \xi$ induces an isomorphism $\Psi : \mathcal{V} \rightarrow \text{AVF}_\omega(S)$. By Proposition 7.1 and Lemma 7.4 up to constant factors Ψ^{-1} sends δ_z, δ_y , and δ_x from Lemma 7.4 to the functions $-z$, $-y$ and $-x$ respectively (which can be checked precisely by a direct computation).

Therefore for $c(z) = -\sum_{i=1}^N c_i z^{i-1}$ the vector field $c(z)\delta_z$ is completely integrable, divergence-free, and $\Psi^{-1}(c(z)\delta_z) = \sum_{i=1}^N c_i z^i$ that is the third nonconstant summand in the Formula (4). The first and the second nonconstant summands can be taken care of by vector fields of form $b(y)\delta_y$ and $a(x)\delta_x$.

Furthermore,

$$i_{[\delta_z, \delta_y]} \omega = d(i_{\delta_z}(i_{\delta_y} \omega)) = d(i_{\delta_z} dy) = dL_{\delta_z}(y) = d(1 + yz) = d(yz).$$

Thus Ψ^{-1} sends the Lie bracket $[z^i \delta_z, y^j \delta_y] \in \text{Lie}_{\text{alg}}^\omega(S)$ to the monomial $z^{i+1} y^{j+1}$. This shows that the last summand in Formula (4) is dual to an element from $\text{Lie}_{\text{alg}}^\omega(S)$. The two remaining nonconstant summands can be treated similarly and thus for any $f \in \mathcal{V}$ one has $\xi = \Psi(f) \in \text{Lie}_{\text{alg}}^\omega(S)$ which yields the desired conclusion. \square

8. APPENDIX: ALGEBRAIC VOLUME FORMS ON HOMOGENEOUS SPACES

In this section we discuss some simple and perhaps known facts about algebraic volume forms. If a smooth affine algebraic variety possesses such a form and does not admit nonconstant invertible regular functions then the form is unique up to a constant factor.

Another well-known fact is that a linear algebraic group G has a left-invariant algebraic volume form (which is simultaneously right-invariant in the case of a reductive G) but a homogeneous space G/R may not have a similar form (see Example 8.5 below). The criterion for existence of such a form on G/R is a straightforward analogue of the criterion about the existence of an invariant Haar measure on a real homogeneous space in terms of modular functions⁷. In order to describe this criterion we need the following.

8.1. Definition. Let G be a linear algebraic group, and N be a subgroup of G , and H be a subgroup of the normalizer of N in G . Consider function $\tilde{\Delta}_{H,N} : H \rightarrow \mathbb{C}^*$ that assigns to each $h \in H$ the determinant $\det \text{ad}_h|_{\mathfrak{n}}$ of the adjoint action of h on the Lie algebra \mathfrak{n} of N (in particular $\tilde{\Delta}_{H,N}$ is a character of H). We say that $\tilde{\Delta}_{H,N}$ the sub-modular function of the pair (H, N) . In the case of $H = N = G$ we call $\tilde{\Delta}_G := \tilde{\Delta}_{G,G}$ the sub-modular function of G .

⁷ Recall that for a real Lie group $G_{\mathbb{R}}$ the modular function $\Delta_{G_{\mathbb{R}}} : G_{\mathbb{R}} \rightarrow \mathbb{R}_+$ is given by $g \rightarrow |\det \text{ad}_g|$ for $g \in G_{\mathbb{R}}$ where ad_g is the adjoint action on the Lie algebra. If $R_{\mathbb{R}}$ is a closed Lie subgroup of $G_{\mathbb{R}}$ then the homogeneous space $G_{\mathbb{R}}/R_{\mathbb{R}}$ has a $G_{\mathbb{R}}$ -invariant Haar measure iff $\Delta_{G_{\mathbb{R}}}|_{R_{\mathbb{R}}} = \Delta_{R_{\mathbb{R}}}$.

8.2. Proposition. *Let G be a connected linear algebraic group, \mathcal{R}_u be the normal subgroup of G associated with the unipotent radical of the Lie algebra of G , H be a maximal reductive subgroup of G , and T be the connected identity component of the center of H . Then $\tilde{\Delta}_G \equiv 1$ iff $\tilde{\Delta}_{T, \mathcal{R}_u} \equiv 1$. In particular, for every connected reductive group its sub-modular function is the trivial character.*

Proof. Let S be a maximal semi-simple subgroup of H . The absence of nontrivial characters on S and \mathcal{R}_u implies that $\tilde{\Delta}_{S, G} \equiv 1$ and $\tilde{\Delta}_{\mathcal{R}_u, G} \equiv 1$. One can present each $g \in G$ as $g = str$ where $s \in S, t \in T$, and $r \in \mathcal{R}_u$. Hence

$$\det \operatorname{ad}_g = \det \operatorname{ad}_s \cdot \det \operatorname{ad}_t \cdot \det \operatorname{ad}_r = \det \operatorname{ad}_t,$$

i.e. $\tilde{\Delta}_G \equiv 1$ iff $\tilde{\Delta}_{T, G} \equiv 1$. Note that also that as a vector space the Lie algebra of G is the direct sum of Lie algebras of H and \mathcal{R}_u . That is, $\tilde{\Delta}_{T, G} = \tilde{\Delta}_{T, H} \tilde{\Delta}_{T, \mathcal{R}_u}$. Since T is in the center of H one has $\tilde{\Delta}_{T, H} \equiv 1$, i.e. $\tilde{\Delta}_{T, G} = \tilde{\Delta}_{T, \mathcal{R}_u}$ which implies the desired conclusion. \square

Before formulating the criterion we need one more simple fact.

8.3. Lemma. *Let $\rho : P \rightarrow X$ be a principal R -bundle where R is a reductive group of dimension m . Suppose that ω_R is an invariant volume form on R and α is a (resp. closed; resp. exact) k -form on X where $0 \leq k \leq n := \dim X$. Then there exist a (resp. closed; resp. exact) R -invariant $(m+k)$ -form α_P on P such that for any open subset $U \subset X$, for which $\rho^{-1}(U)$ is naturally isomorphic to $U \times R$, the restriction of α_P to $\rho^{-1}(U)$ coincides with $\alpha \times \omega_R$.*

Proof. Consider an open covering $\{U_i\}$ of X such that $\rho^{-1}(U_i)$ is naturally isomorphic to $U_i \times R$. The structure of direct product enables us to consider an R -invariant form $\omega_i = \alpha \times \omega_R$ on every $\rho^{-1}(U_i)$. The transition isomorphism over $U_i \cap U_j$ is of form $(u, r) \rightarrow (u, g(u)r)$ where $u \in U_i \cap U_j$ and $r, g(u) \in R$. Since ω_R is an invariant form we see that ω_i and ω_j agree on $\rho^{-1}(U_i \cap U_j)$ which implies the desired conclusion. \square

8.4. Proposition. *Let $X = G/R$ be a homogeneous space of left cosets where G is a linear algebraic group and R is a closed reductive subgroup of G . Then $\tilde{\Delta}_R \equiv \tilde{\Delta}_G|_R$ if and only if there exists an algebraic volume form ω_X on X invariant under the action of G generated by left multiplication. In particular, in the case of connected reductive G and R such a volume form ω_X always exists (by Proposition 8.2).*

Proof. Choose a left-invariant volume form ω on G and left-invariant vector fields ν_1, \dots, ν_m on G . These fields are tangent to all fibers of the natural projection $p : G \rightarrow X$. Consider the left-invariant form $\omega_X = \iota_{\nu_1} \circ \dots \circ \iota_{\nu_m}(\omega)$. By construction it can be viewed as a non-vanishing form on vectors from the pull-back of the tangent bundle TX to G . To see that it is actually a volume form on X we have to show that it is invariant under multiplication by any element $r \in R$. Such multiplication generates an automorphism of TG that sends vectors tangent (and, therefore, transversal) to fibers of p to similar vectors. Hence it transforms ω_X into $f_r \omega_X$ where f_r is an invertible function

on G . Since modulo \mathbb{C}^* the group of invertible functions is a discrete set (more precisely, it is $H^1(G, \mathbb{Z})$; e.g., see [8]) and $f_e \equiv 1$ (where e is the identity of G) we see that f_r is a nonzero constant for every r . Hence the map $r \rightarrow f_r$ yields a homomorphism from R into \mathbb{C}^* . Since ω is left-invariant we see that $r^{-1} \circ \omega_X \circ r = f_r \omega_X$ (where $r^{-1} \circ \omega_X \circ r$ is the image of ω_X under conjugation by r). Consider the last equality at $e \in G$ treating ω_X as the result of the evaluation of ω at the wedge product $\mu = \nu_1 \wedge \dots \wedge \nu_m$. Note that conjugation by r transforms ω at e into $\tilde{\Delta}_G(r)\omega$ while transforming μ into $(\tilde{\Delta}_R(r))^{-1}\mu$ (because μ is dual to ω_R). Hence $f_r(e) = \tilde{\Delta}_G(r)/\Delta_R(r)$. Since $f_r(g)$ is independent from $g \in G$ we see that ω_X is invariant under right multiplication by elements of R iff $\tilde{\Delta}_R \equiv \tilde{\Delta}_G|_R$.

The other direction: by Lemma 8.3 in the presence of an algebraic volume form ω_X on X consider the volume form ω' on G such that for every open $U \subset X$ with $p^{-1}(U) \simeq U \times R$ the restriction of ω' to $p^{-1}(U)$ coincides with $\omega_X \times \omega_R$. Note that up to a constant factor $\iota_{\nu_1} \circ \dots \circ \iota_{\nu_m}(\omega')$ coincides with ω_X since $\iota_{\nu_1} \circ \dots \circ \iota_{\nu_m}(\omega_R)$ is constant and $\iota_{\nu_1} \circ \dots \circ \iota_{\nu_m}(\omega_X \times \omega_R) = (\iota_{\nu_1} \circ \dots \circ \iota_{\nu_m}(\omega_R))\omega_X$.

Thus we can suppose that $\omega_X = \iota_{\nu_1} \circ \dots \circ \iota_{\nu_m}(\omega')$. Note also that ω' is left-invariant provided ω_X is left-invariant, i.e. $\omega' = \omega$. That is, the relation between the left-invariant form ω on G and ω_X is the same as in the first part of the proof which yields the desired conclusion. \square

8.5. Example. Unlike in Proposition 8.2 for a non-connected reductive group the submodular function may differ from a trivial character. Consider for instance a nontrivial extension N of \mathbb{C}^* by \mathbb{Z}_2 , i.e. for every $z \in \mathbb{C}^*$ and $g \in N \setminus \mathbb{C}^*$ we have $gzg^{-1} = z^{-1}$. Then the value $\tilde{\Delta}_N$ on one connected component of N is -1 . In particular, if one treats N as the normalizer of a maximal torus in SL_2 then SL_2/N does not possess a left invariant volume form by Proposition 8.4. Furthermore, SL_2/N does not admit nonconstant invertible regular functions and therefore any two algebraic volume forms must be proportional which implies the absence of algebraic volume forms of SL_2/N .

It is worth mentioning the following cohomological interpretation of volume forms.

8.6. Proposition. *Let X be a homogeneous space of a connected reductive group G and ω be an G -invariant volume form on X . Suppose that $\dim X = n$ and $H^n(X) \neq 0$ (i.e. $H^n(X, \mathbb{C}) = \mathbb{C}$). Then De Rham homomorphism sends ω to a generator of $H^n(X, \mathbb{C})$.*

Proof. Suppose that under De Rham isomorphism a closed form $f\omega$ corresponds to a generator α of $H^n(X, \mathbb{Z})$ and that $K = G_{\mathbb{R}}$ is a maximal compact subgroup of G (i.e. G is the complexification $K_{\mathbb{C}}$ of K). Since the K -action on X induces the identical automorphism of $H^n(G, \mathbb{Z})$ we see that $(f \circ k)\omega$ corresponds again to α where $f \circ k$ is the image of f under the action of $k \in K$. Hence $g\omega$ corresponds to α where $g = \int_K (f \circ k)\mu$ and μ is the invariant Haar measure on K . Since g is K -invariant it must be a constant function on X which implies the desired conclusion. \square

8.7. Corollary. *Let an affine algebraic manifold X without nonconstant regular invertible functions possess an algebraic volume form ω such that under De Rham homomorphism ω corresponds to the zero element of $H^n(X, \mathbb{C}) = \mathbb{C}$. Then X cannot be a homogeneous space of any nontrivial reductive group.*

8.8. Remark. It was shown in [6] that any variety $X_{m,1} = \{x^m v - yu = 1\} \subset \mathbb{C}_{x,y,u,v}^4$ with $m \geq 2$ is diffeomorphic (as a real manifold) but not isomorphic to $X_{1,1} \simeq SL_2$ ⁸ because the unique (up to a constant factor) volume form $\omega_m = x^{-m} dx \wedge dy \wedge du$ on $X_{m,1}$ is exact ($\omega_m = d\tau$ where $\tau = \frac{dy \wedge du}{(1-m)x^{m-1}}$). Corollary 8.7 enables us to tell now more: $X_{m,1}$ is not isomorphic to a homogeneous space of a nontrivial reductive group.

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⁸Such varieties $X_{m,1}$ with $m \geq 2$ are examples of “complex exotic affine 3-spheres” since SL_2 is isomorphic to $\{x^2 + y^2 + u^2 + v^2 = 1\} \subset \mathbb{C}^4$.

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